# CRITICAL HEEGAARD SURFACES OBTAINED BY AMALGAMATION

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ABSTRACT. Critical surfaces are defined by Bachman as topological index 2 surfaces, generalizing incompressible surfaces and strongly irreducible surfaces. In this paper we give a condition to obtain critical Heegaard surfaces by amalgamation. As a special case, we obtain critical Heegaard surfaces by boundary stabilization. It gives critical Heegaard surfaces of non-minimal genus, for 3-manifolds which do not admit distinct Heegaard splittings (up to isotopy).

## 1. INTRODUCTION

Let $S$ be a closed orientable separating surface in an irreducible 3-manifold $M$, dividing $M$ into two submanifolds $V$ and $W$. Define the *disk complex* $\mathcal{D}_S$ as follows.

- Vertices of $\mathcal{D}_S$ are isotopy classes of compressing disks for $S$.
- A collection of $k + 1$ distinct vertices constitute a $k$-cell if there are mutually disjoint representatives.

By an abuse of terminology, we sometimes identify a vertex with some representative compressing disk of the vertex. Let $\mathcal{D}_S(V)$ and $\mathcal{D}_S(W)$ be the subcomplexes of $\mathcal{D}_S$ spanned by compressing disks in $V$ and $W$ respectively. In this paper, we focus only on vertices and edges of $\mathcal{D}_S(V)$, $\mathcal{D}_S(W)$, and $\mathcal{D}_S$.

If $S$ is an incompressible surface, then $\mathcal{D}_S$ is empty, and vice versa. We say that $S$ is *strongly irreducible* if both $\mathcal{D}_S(V)$ and $\mathcal{D}_S(W)$ are non-empty, and $\mathcal{D}_S(V)$ is not connected to $\mathcal{D}_S(W)$ in $\mathcal{D}_S$. Strongly irreducible surfaces are proved to be useful to analyze the Heegaard structure and topology of 3-manifolds. For example, if a minimal genus Heegaard surface is not strongly irreducible, then the manifold contains an incompressible surface [5].

An incompressible surface and a strongly irreducible surface can be regarded as topological analogues of an index 0 minimal surface and an index 1 minimal surface respectively. As a generalization of this idea, Bachman has defined a notion of *critical surface* [1], [2], [3], which can be regarded as a topological index 2 minimal surface. It is equivalent to that $\pi_1(\mathcal{D}_S)$ is non-trivial.

**Definition 1.1.** *A surface $S$ is critical if vertices of $\mathcal{D}_S$ can be partitioned into two non-empty sets $C_0$ and $C_1$.*

(1) *For each $i = 0, 1$, there is at least one pair of compressing disks $D_i \in \mathcal{D}_S(V) \cap C_i$ and $E_i \in \mathcal{D}_S(W) \cap C_i$ such that $D_i \cap E_i = \emptyset$.*

(2) *If $D \in \mathcal{D}_S(V) \cap C_i$ and $E \in \mathcal{D}_S(W) \cap C_{1-i}$, then $D \cap E \neq \emptyset$ for any*
    *representative disks. In other words, $D$ and $E$ are not joined by an*
    *edge.*

Critical surfaces behave in some way similarly as incompressible surfaces
and strongly irreducible surfaces do. For example, if an irreducible mani-
fold contains an incompressible surface and a critical surface, then the two
surfaces can be isotoped so that any intersection loop is essential on both
surfaces. In fact, this is true for all topologically minimal surfaces [3].

Topological index theory, including critical surfaces, is a relatively new
theory pioneered by Bachman. One can ask how large the class of critical
surfaces is. In [1], it was shown that if a manifold which does not con-
tain incompressible surfaces has two distinct strongly irreducible Heegaard
splittings, then the minimal genus common stabilization of the two split-
tings is critical. (In fact the original definition of criticality in [1] is slightly
stronger than Definition 1.1. Hence the above mentioned result still holds
with respect to Definition 1.1.)

In this paper, we show that some critical Heegaard surfaces can be ob-
tained by amalgamating two strongly irreducible Heegaard splittings.

**Theorem 1.2.** *Let $X \cup_S Y$ be an amalgamation of two strongly irreducible*
*Heegaard splittings $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ along homeomorphic boundary*
*components of $\partial_- V_1$ and $\partial_- V_2$. See Figure 2. Assume that $V_2$ is constructed*
*from $\partial_- V_2 \times I$ by attaching only one 1-handle. If there exist essential disks*
*$D_1 \subset W_1$ and $D_2 \subset W_2$ which persist into disjoint essential disks in $Y$ and*
*$X$ respectively, then $S$ is critical.*

As a special case of Theorem 1.2, we obtain critical Heegaard surfaces by
boundary stabilization—an amalgamation with a standard type 2 Heegaard
splitting of (surface) $\times I$ [8]. This gives plenty of new critical Heegaard
surfaces of non-minimal genus, also for 3-manifolds which do not admit
distinct Heegaard splittings. In other words, it says the existence of a critical
Heegaard surface of non-minimal genus for a 3-manifold admitting a unique
minimal genus Heegaard splitting.

**Theorem 1.3.** *If a strongly irreducible Heegaard surface of a 3-manifold*
*with boundary admits a disjoint pair of a vertical annulus and an essential*
*disk on opposite sides, then the Heegaard splitting obtained from it by a*
*boundary stabilization is critical.*

*Remark* 1.4. Other examples of critical Heegaard surfaces, including high
index topologically minimal surfaces, are constructed in [4]. On the other
hand, unstabilized examples of critical Heegaard surfaces are shown in [6].

Our main theorem is Theorem 1.2, but we consider Theorem 1.3 first
because it is easier. In Section 2, we define boundary stabilization of a
Heegaard splitting. In Section 3, we give a proof of Theorem 1.3. In Section
4, we define amalgamation of two Heegaard slittings. In Section 5, we give
a proof of Theorem 1.2.

## 2. Boundary stabilization

A *compression body* $V$ is a 3-manifold obtained from a closed surface $S$ by attaching some 2-handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2-sphere boundary components with 3-balls. The surface $S \times \{1\}$ is denoted by $\partial_+ V$ and $\partial V - \partial_+ V$ is denoted by $\partial_- V$. If $\partial_- V = \emptyset$, then $V$ is called a *handlebody*. There is also a dual description for compression body. When $\partial_- V \neq \emptyset$, $V$ can be obtained from $\partial_- V \times I$ by attaching dual 1-handles to $\partial_- V \times \{1\}$. For a 3-manifold $M$, a *Heegaard splitting* $V \cup_S W$ is a decomposition of $M$ into two compression bodies $V$ and $W$, where $\partial_+ V = \partial_+ W = S$ and $\partial M = \partial_- V \cup \partial_- W$. It is known that every compact 3-manifold admits Heegaard splittings.

For a Heegaard splitting $V \cup_S W$, let $\alpha$ be a properly embedded arc in $W$ which is parallel to an arc in $S$. We add $N(\alpha)$ to $V$ and remove it from $W$. Then we get a new Heegaard splitting $V' \cup_{S'} W'$ with the genus increased by one. This is called a *stabilization* of $V \cup_S W$. Conversely, for a stabilized Heegaard splitting $V' \cup_{S'} W'$ there exist essential disks $D \subset V'$ and $E \subset W'$ such that $|D \cap E| = 1$, and $V' \cup_{S'} W'$ can be *destabilized* to a lower genus Heegaard splitting.

Let $g(\cdot)$ denote the genus of a surface. For Heegaard splittings under consideration in Section 2 and 3, we assume that $\partial_- V \neq \emptyset$ and $g(\partial_- V) < g(\partial_+ V)$ and $W$ is a handlebody.

A *complete meridian disk system* $\{D_i\}$ for $V$ is a collection of disjoint essential disks with $\partial D_i \subset \partial_+ V$ such that cutting $V$ along $\bigcup D_i$ results in $\partial_- V \times I$. A *vertical annulus* $A$ in $V$ is a properly embedded essential annulus with one boundary component in $\partial_- V$ and the other boundary component in $\partial_+ V$. By a standard argument, there exists a complete meridian disk system for $V$ which is disjoint from $A$. Hence there exists a dual description $V = (\partial_- V \times [0,1]) \cup$ 1-handles in which $A$ inherits the product structure.

Take a vertical arc $\gamma$ in $A$. A tubular neighborhood $N(\gamma)$ can be regarded as a 1-handle connecting $W$ to $\partial_- V \times [0, \frac{1}{2}]$. Compression bodies $V'$ and $W'$ of a *boundary stabilization* $V' \cup_{S'} W'$ of $V \cup_S W$ are defined as follows.

- $W' = W \cup N(\gamma) \cup (\partial_- V \times [0, \frac{1}{2}])$
- $V' = \mathrm{cl}(M - W')$

Clearly $W'$ is a compression body. We can see that $V'$ is homeomorphic to (punctured $\partial_- V) \times [\frac{1}{2}, 1] \cup$ 1-handles. So $V' \cup_{S'} W'$ becomes a Heegaard splitting and $g(S')$ is equal to $g(S) + g(\partial_- V)$.

Boundary stabilization gives a way to obtain a higher genus Heegaard splitting from a given Heegaard splitting of a 3-manifold with boundary. There are many cases that the boundary stabilized surfaces can be destabilized, although it is not clear whether it can always be destabilized. See ([7], Section 5). For example, a boundary stabilization of a splitting admitting a pair of a vertical annulus and an essential disk intersecting in one point, which is called $\gamma$-*primitive* in [7], results in a stabilized Heegaard splitting. It is easy to see that $\gamma$-primitive Heegaard splitting satisfies the condition of Theorem 1.3.

## 3. Partition of disk complex

In this section, we give a proof of Theorem 1.3. Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting of a 3-manifold with boundary admitting a vertical annulus $A \subset V$ and an essential disk $D \subset W$ with $A \cap D = \emptyset$. Take a vertical arc $\gamma \subset A$ and boundary stabilize $V \cup_S W$ to get $V' \cup_{S'} W'$ as in Section 2. See Figure 1. We will show that $S'$ is a critical Heegaard surface of $M$.
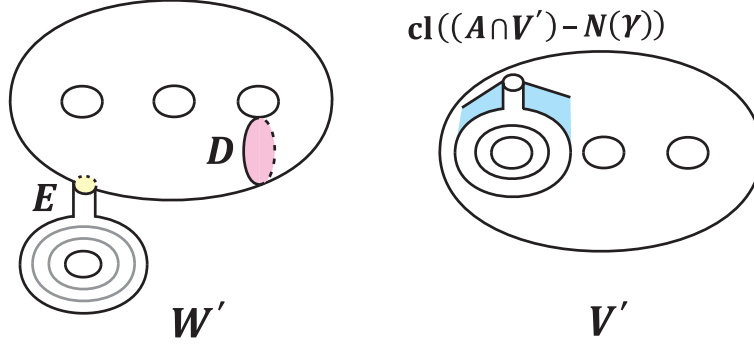


FIGURE 1. Boundary stabilization

Let $E$ be a meridian disk of $W'$ corresponding to the 1-handle $N(\gamma)$. Define a partition of vertices of the disk complex $\mathcal{D}_{S'} = C_0 \mathbin{\dot\cup} C_1$ as follows.

(1) Let $\mathcal{D}_{S'}(V') \cap C_0$ be essential disks in $V'$ that are disjoint from $E$.
(2) Let $\mathcal{D}_{S'}(W') \cap C_0$ be a singleton $\{E\}$.
(3) Let $\mathcal{D}_{S'}(V') \cap C_1$ be $\mathcal{D}_{S'}(V') - (\mathcal{D}_{S'}(V') \cap C_0)$.
(4) Let $\mathcal{D}_{S'}(W') \cap C_1$ be $\mathcal{D}_{S'}(W') - \{E\}$.

Since $g(\partial_- V) < g(\partial_+ V)$, there are essential disks in $V$ disjoint from $\gamma$. Hence there are essential disks in $V'$ that are disjoint from $E$. This means that $C_0$ contains disjoint compressing disks for $S'$ on opposite sides.

Since $\gamma$ is contained in the annulus $A$, $\mathrm{cl}((A \cap V') - N(\gamma))$ is an essential disk in $V'$ intersecting $E$ (minimally) in two points, so it belongs to $\mathcal{D}_{S'}(V') \cap C_1$. The essential disk $D$ persists as an essential disk in $W'$ and belongs to $\mathcal{D}_{S'}(W') \cap C_1$. By assumption, $\mathrm{cl}((A \cap V') - N(\gamma))$ and $D$ are disjoint. So $C_1$ also contains disjoint compressing disks for $S'$ on opposite sides. Hence the first condition of Definition 1.1 is satisfied.

By definition, any disk in $\mathcal{D}_{S'}(V') \cap C_1$ intersects $E$. Now it remains to show that any disk in $\mathcal{D}_{S'}(V') \cap C_0$ intersects any disk in $\mathcal{D}_{S'}(W') \cap C_1$.

Let $D_0 \in \mathcal{D}_{S'}(V') \cap C_0$ be an essential disk in $V'$ that is disjoint from $E$. Note that $V'$ is homeomorphic to (punctured $\partial_- V) \times [\frac{1}{2}, 1] \cup$ 1-handles, and $\partial E$ corresponds to the puncture. So we can see that $\partial D_0$ should be contained in $\partial_+ V \cap S'$. This means that $D_0$ can be regarded as an essential disk in $V$. Let $E_1 \in \mathcal{D}_{S'}(W') \cap C_1$ be an essential disk in $W'$ which is not isotopic to $E$.

*Claim.* $D_0 \cap E_1 \neq \emptyset$.

*Proof.* Suppose to the contrary that $D_0 \cap E_1 = \emptyset$. We assume that $E_1$ is chosen so that the number of components of intersection $|E \cap E_1|$ is minimal

up to isotopy of $E_1$, satisfying $D_0 \cap E_1 = \emptyset$. Since a circle component of intersection can be removed by a standard innermost disk argument, we can assume that there is no circle component of intersection in $E \cap E_1$ Hence $E \cap E_1$ consists of arc components. Let $\alpha$ be an outermost arc component in $E_1$ and $\Delta$ be the corresponding outermost disk in $E_1$. Let $\beta = cl(\partial\Delta - \alpha)$.

The disk $E$ separates $W'$ into $W$ and a manifold homeomorphic to $\partial_- V \times [0, \frac{1}{2}]$. Suppose $\Delta$ is contained in $\partial_- V \times [0, \frac{1}{2}]$. Surger $E$ along $\Delta$, and let $E'$ and $E''$ be the two resulting disks. Then $E'$ is an inessential disk in $W'$ and $E''$ is isotopic to $E$, or vice versa. In any case we can reduce $|E \cap E_1|$, a contradiction.

So $\Delta$ is contained in $W$. Let $E'$ be one of the disk obtained by surgery of $E$ along $\Delta$. By minimality of $|E \cap E_1|$ again, $E'$ is neither inessential in $W'$ nor isotopic to $E$. Hence $E'$ can be regarded as an essential disk in $W$. Since $V \cup_S W$ is a strongly irreducible Heegaard splitting, $D_0 \cap E' \neq \emptyset$. We can observe that

$$D_0 \cap E' = \partial D_0 \cap \partial E' = \partial D_0 \cap \beta \subset \partial D_0 \cap \partial E_1$$

However, $D_0 \cap E_1 = \emptyset$ by assumption, and this is a contradiction. $\qquad \square$

Hence the partition of the disk complex $\mathcal{D}_{S'} = C_0 \,\dot{\cup}\, C_1$ satisfies the criticality. This completes the proof of Theorem 1.3.

## 4. AMALGAMATION

In this section we give a definition of amalgamation, which was first introduced in [9]. Boundary stabilization discussed in Section 2 and 3 is an amalgamation with a standard type 2 Heegaard splitting of (surface)$\times I$. Let $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ be Heegaard splittings of 3-manifolds $M_1$ and $M_2$ respectively. Suppose $M_1$ and $M_2$ are glued together along some homeomorphic boundary components $F_1 \subset \partial_- V_1$ and $F_2 \subset \partial_- V_2$. Let $M = M_1 \cup_F M_2$ be the resulting manifold and $F$ be the image of $F_1$ and $F_2$ in $M$.
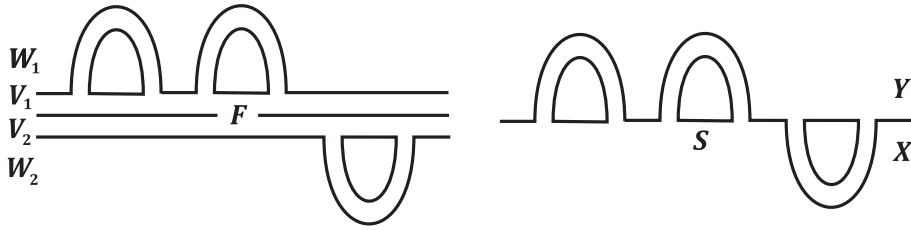


FIGURE 2. Amalgamation

By dual description of compression body, $V_1$ can be regarded as $(\partial_- V_1 \times [0, 1]) \cup$ 1-handles and $V_2$ can be regarded as $(\partial_- V_2 \times [0, 1]) \cup$ 1-handles. Now collapse $(F_1 \cup F_2) \times [0, 1]$ to $F$ and regard the 1-handles of $V_1$ and $V_2$ are attached to $F$. We assume that the disks where the 1-handles are attached are mutually disjoint. Let $X = W_2 \cup$ (1-handles in $V_1$) and $Y = W_1 \cup$ (1-handles in $V_2$). See Figure 2. It is obvious that $X$ and $Y$ are compression bodies (or handlebodies). The Heegaard splitting $M = X \cup_S Y$ is called an *amalgamation* of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$. By construction, $g(S) =$

$g(S_1) + g(S_2) - g(F)$. We can see that essential disks in $W_1$ and $W_2$ persist into essential disks in $Y$ and $X$ respectively.

## 5. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. The underlying idea is same with the proof of Theorem 1.3.

Note that $V_2$ is constructed from $\partial_- V_2 \times I$ by attaching only one 1-handle. Let $E$ be a meridian disk of $Y$ corresponding to the 1-handle. Define a partition of vertices of the disk complex $\mathcal{D}_S = C_0 \dot\cup C_1$ as follows.

(1) Let $\mathcal{D}_S(X) \cap C_0$ be essential disks in $X$ that are disjoint from $E$.
(2) Let $\mathcal{D}_S(Y) \cap C_0$ be a singleton $\{E\}$.
(3) Let $\mathcal{D}_S(X) \cap C_1$ be $\mathcal{D}_S(X) - (\mathcal{D}_S(X) \cap C_0)$.
(4) Let $\mathcal{D}_S(Y) \cap C_1$ be $\mathcal{D}_S(Y) - \{E\}$.

The co-core disks of 1-handles of $V_1$ persist into essential disks in $X$ disjoint from $E$. Hence $C_0$ contains disjoint compressing disks for $S$ on opposite sides.

The two disks $D_1$ and $D_2$ persist into disjoint essential disks in $Y$ and $X$ respectively. Since $V_2 \cup_{S_2} W_2$ is strongly irreducible, $D_2$ intersects $E$. Hence after amalgamation, $D_1$ and $D_2$ represent elements in $\mathcal{D}_S(Y) \cap C_1$ and $\mathcal{D}_S(X) \cap C_1$ respectively. So $C_1$ also contains disjoint compressing disks for $S$ on opposite sides. The first condition of Definition 1.1 is now satisfied.

By definition of the partition, any disk in $\mathcal{D}_S(X) \cap C_1$ intersects $E$. Now it remains to show that any disk in $\mathcal{D}_S(X) \cap C_0$ intersects any disk in $\mathcal{D}_S(Y) \cap C_1$.
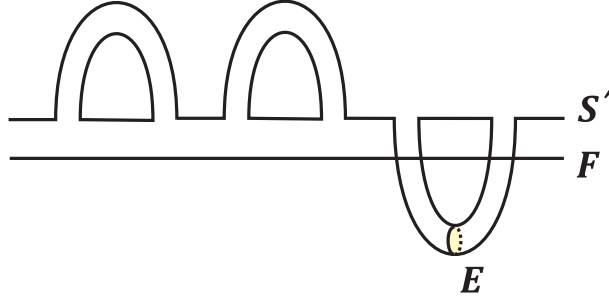


FIGURE 3.

Let $D_0 \in \mathcal{D}_S(X) \cap C_0$ be an essential disk in $X$ that is disjoint from $E$. The loop $\partial D_0$ is contained in a surface $S'$ homeomorphic to twice punctured $S_1$. Take a copy of the surface $F$ so that the region between $S'$ and $F$ is homeomorphic to $V_1$ as in the Figure 3. Since both $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ are strongly irreducible, $F$ is incompressible in $M$. Then by standard argument, we can assume that $D_0 \cap F = \emptyset$. Hence $D_0$ can be regarded as an essential disk in $V_1$. Let $E_1 \in \mathcal{D}_S(Y) \cap C_1$ be an essential disk in $Y$ which is not isotopic to $E$.

*Claim.* $D_0 \cap E_1 \neq \emptyset$.

*Proof.* Suppose to the contrary that $D_0 \cap E_1 = \emptyset$. We assume that $E_1$ is chosen so that the number of components of intersection $|E \cap E_1|$ is minimal

up to isotopy of $E_1$, keeping that $D_0 \cap E_1 = \emptyset$. Since a circle component of intersection can be removed by a standard innermost disk argument, we can assume that there is no circle component of intersection in $E \cap E_1$. Hence $E \cap E_1$ consists of arc components. Let $\alpha$ be an outermost arc component in $E_1$ and $\Delta$ be the corresponding outermost disk in $E_1$. Let $\beta = cl(\partial\Delta - \alpha)$.

The disk $E$ cuts $Y$ into a manifold homeomorphic to $W_1$. Surger $E$ along $\Delta$, and let $E'$ be one of the two resulting disks. By minimality of $|E \cap E_1|$, $E'$ is neither inessential in $W_1$ nor isotopic to $E$. Hence $E'$ can be regarded as an essential disk in $W_1$. Since $V_1 \cup_{S_1} W_1$ is a strongly irreducible Heegaard splitting, $D_0 \cap E' \neq \emptyset$. We can observe that

$$D_0 \cap E' = \partial D_0 \cap \partial E' = \partial D_0 \cap \beta \subset \partial D_0 \cap \partial E_1$$

However, $D_0 \cap E_1 = \emptyset$ by assumption, and this is a contradiction. $\qquad\square$

Hence the partition of the disk complex $\mathcal{D}_S = C_0 \stackrel{.}{\cup} C_1$ satisfies the criticality. This completes the proof of Theorem 1.2.

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